# G.C.E. (A.L.) Examination - 2019
# 10 - Combined Mathematics I
# (New Syllabus)

## Distribution of Marks

**Paper I**

**Part A : 10 × 25 = 250**

**Part B : 05 × 150 = 750**

**Total = 1000 / 10**

**Paper I Final Mark = 100**

## Common Techniques of Marking Answer Scripts.

It is compulsory to adhere to the following standard method in marking answer scripts and entering marks into the mark sheets.

1. Use a red color ball point pen for marking. (Only Chief/Additional Chief Examiner may use a mauve color pen.)
2. Note down Examiner's Code Number and initials on the front page of each answer script.
3. Write off any numerals written wrong with a clear single line and authenticate the alterations with Examiner's initials.
4. Write down marks of each subsection in a $\triangle$ and write the final marks of each question as a rational number in a $\square$ with the question number. Use the column assigned for Examiners to write down marks.

**Example:** **Question No. 03**

(i) ..................................................... √ $\frac{4}{5}$
.....................................................
.....................................................

(ii) ..................................................... √ $\frac{3}{5}$
.....................................................
.....................................................

(iii) ..................................................... √ $\frac{3}{5}$
.....................................................
.....................................................

03 (i) $\frac{4}{5}$ + (ii) $\frac{3}{5}$ + (iii) $\frac{3}{5}$ = $\boxed{\frac{10}{15}}$

### MCQ answer scripts: (Template)

1. Marking templets for G.C.E.(A/L) and GIT examination will be provided by the Department of Examinations itself. Marking examiners bear the responsibility of using correctly prepared and certified templates.
2. Then, check the answer scripts carefully. If there are more than one or no answers Marked to a certain question write off the options with a line. Sometimes candidates may have erased an option marked previously and selected another option. In such occasions, if the erasure is not clear write off those options too.
3. Place the template on the answer script correctly. Mark the right answers with a '√' and the wrong answers with a 'X' against the options column. Write down the number of correct answers inside the cage given under each column. Then, add those numbers and write the number of correct answers in the relevant cage.

## Structured essay type and assay type answer scripts:

1. Cross off any pages left blank by candidates. Underline wrong or unsuitable answers. Show areas where marks can be offered with check marks.

2. Use the right margin of the overland paper to write down the marks.

3. Write down the marks given for each question against the question number in the relevant cage on the front page in two digits. Selection of questions should be in accordance with the instructions given in the question paper. Mark all answers and transfer the marks to the front page, and write off answers with lower marks if extra questions have been answered against instructions.

4. Add the total carefully and write in the relevant cage on the front page. Turn pages of answer script and add all the marks given for all answers again. Check whether that total tallies with the total marks written on the front page.

## Preparation of Mark Sheets.

Except for the subjects with a single question paper, final marks of two papers will not be calculated within the evaluation board this time. Therefore, add separate mark sheets for each of the question paper. Write paper 01 marks in the paper 01 column of the mark sheet and write them in words too. Write paper II Marks in the paper II Column and wright the relevant details. For the subject 51 Art, marks for Papers 01, 02 and 03 should be entered numerically in the mark sheets.

***

**1.** Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n}(2r-1) = n^2$ for all $n \in \mathbb{Z}^+$.

For $n = 1$, L.H.S. $= 2 \times 1 - 1 = 1$ and R.H.S. $= 1^2 = 1$ 5

$\therefore$ The result is true for $n = 1$.

Take any $p \in \mathbb{Z}^+$ and assume that the result is true for $n = p$.

i.e. $\sum_{r=1}^{p} (2r-1) = p^2$. 5

Now $\sum_{r=1}^{p+1} (2r-1) = \sum_{r=1}^{p} (2r-1) + \big(2(p+1) - 1\big)$ 5

$= p^2 + (2p + 1)$

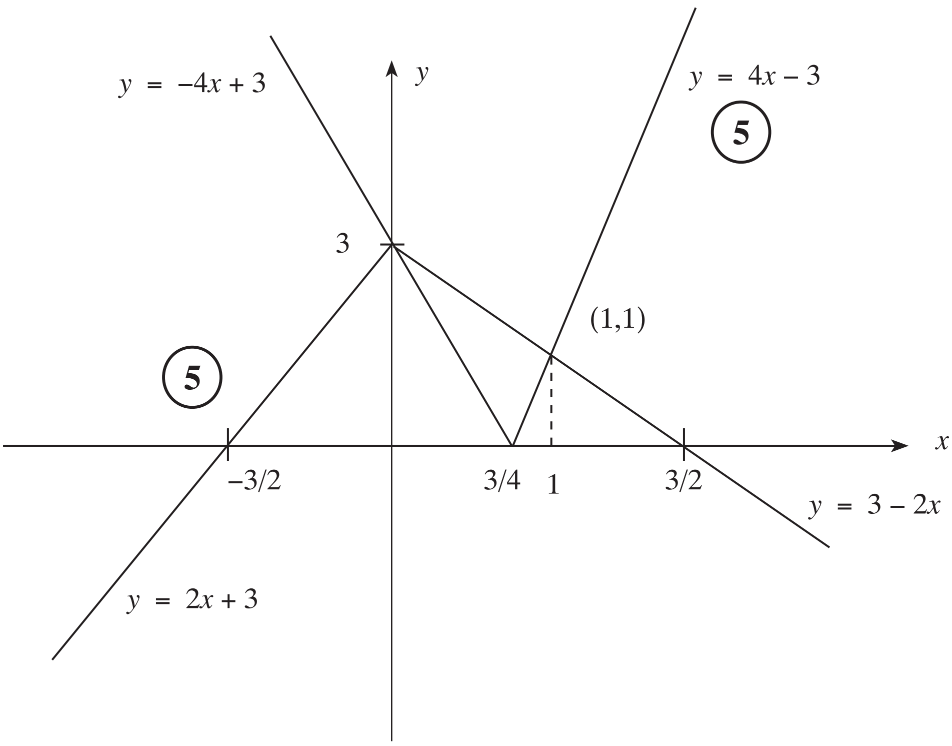
$= (p+1)^2$. 5

Hence, if the result is true for $n = p$, then it is true for $n = p + 1$. We have already proved that the result is true for $n = 1$.

Hence, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$. 5

25

**2.** Sketch the graphs of $y=|4x-3|$ and $y=3-2|x|$ in the same diagram.
**Hence or otherwise**, find all real values of $x$ satisfying the inequality $|2x-3|+|x|<3$.




At the point of intersections of the graphs

$$4x - 3 = 3 - 2x \Rightarrow x = 1 \quad 5$$

$$-4x + 3 = 3 + 2x \Rightarrow x = 0$$

From the graphs, we have,

$$|4x-3| < 3 - 2|x| \quad \Leftrightarrow \quad 0 < x < 1$$

$$\therefore \; |4x-3| + |2x| < 3 \quad \Leftrightarrow \quad 0 < x < 1$$

Replacing $x$ by $\frac{x}{2}$, we get

$$|2x-3| + |x| < 3 \quad \Leftrightarrow \quad 0 < x < 2. \quad 5$$

Hence, the set of all values of $x$ satisfying

$|2x-3| + |x| < 3$ is $\{x : 0 < x < 2\}$. 5

**25**

Aliter

For the graphs 5 + 5 , as before.

Aliter for values of $x$

$|2x-3| + |x| < 3$

Case (i) $x \leq 0$ :

Then $|2x-3| + |x| < 3 \Leftrightarrow -2x+3-x < 3$

$\Leftrightarrow 3x > 0$

$\Leftrightarrow x > 0$

Hence, in this case, no solutions exist.

Case (ii) $0 < x \leq \frac{3}{2}$

Then $|2x-3| + |x| < 3 \Leftrightarrow -2x+3+x < 3$

$\Leftrightarrow x > 0$

Hence, in this case, the solutions are the values of $x$ satisfying $0 < x \leq \frac{3}{2}$.

Case (iii) $x > \frac{3}{2}$

Then $|2x-3| + |x| < 3 \Leftrightarrow 2x-3+x < 3$

$\Leftrightarrow 3x < 6$

$\Leftrightarrow x < 2$

Hence, in this case, the solutions are the values of $x$ satisfying $\frac{3}{2} < x < 2$.

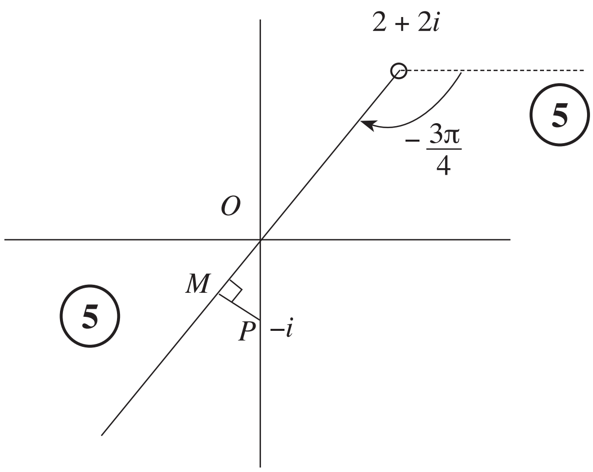
| | |
|---|---|
| All 3 cases with correct solutions | 10 |
| Any 2 cases with correct solutions | 5 |

Hence, over all, the solutions are values of $x$ satisfying $0 < x < 2$. 5

$\boxed{25}$

**3.** Sketch, in an Argand diagram, the locus of the points that represent complex numbers $z$ satisfying $\text{Arg}(z-2-2i) = -\frac{3\pi}{4}$.

**Hence or otherwise**, find the minimum value of $|i\bar{z}+1|$ such that $\text{Arg}(z-2-2i) = -\frac{3\pi}{4}$.




Note that

$|i\bar{z}+1| = |i(\bar{z}-i)| = |\bar{z}-i| = |\overline{z+i}|$

$= |z+i|$

$= |z-(-i)|$ 5

Hence, the minimum of $|i\bar{z}+1|$ is equal to PM. 5

Now, $\text{PM} = 1 \, . \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ . 5

**25**

**4.** Show that the coefficient of $x^6$ in the binomial expansion of $\left(x^3 + \frac{1}{x^2}\right)^7$ is 35.

Show also that there **does not exist** a term independent of $x$ in the above binomial expansion.

$$\left(x^3 + \frac{1}{x^2}\right)^7 = \sum_{r=0}^{7} {}^7C_r \, (x^3)^r \left(\frac{1}{x^2}\right)^{7-r} \quad 5$$

$$= \sum_{r=0}^{7} {}^7C_r \, x^{5r-14}$$

$x^6$ : $5r - 14 = 6 \Leftrightarrow r = 4.$ 5

$\therefore$ The coefficient of $x^6 = {}^7C_4 = 35$ 5

For the above expansion to have a term independent of $x$, we must have

$5r - 14 = 0.$ 5

This is not possible as $r \in \mathbb{Z}^+$. 5

$\boxed{25}$

**5.** Show that $\lim\limits_{x \to 3} \dfrac{\sqrt{x-2}-1}{\sin(\pi(x-3))} = \dfrac{1}{2\pi}$.

$$\lim_{x \to 3} \frac{\sqrt{x-2}-1}{\sin(\pi(x-3))} = \lim_{x \to 3} \frac{\sqrt{x-2}-1}{\sin(\pi(x-3))} \cdot \frac{(\sqrt{x-2}+1)}{(\sqrt{x-2}+1)} \quad 5$$

$$= \lim_{x-3 \to 0} \frac{x-3}{\sin(\pi(x-3))} \cdot \lim_{x \to 3} \frac{1}{(\sqrt{x-2}+1)} \quad 5$$

$$= \lim_{x-3 \to 0} \frac{1}{\dfrac{\sin(\pi(x-3))}{\pi(x-3)}} \cdot \frac{1}{\pi} \cdot \frac{1}{2} \quad 5$$

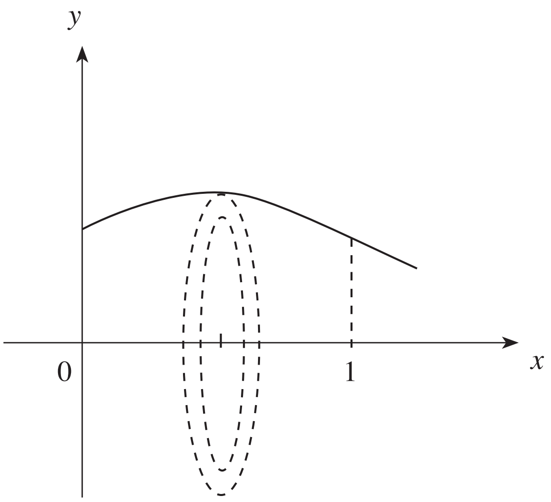
$$= 1 \cdot \frac{1}{\pi} \cdot \frac{1}{2}$$

5

$$= \frac{1}{2\pi} \quad 5$$

**25**

**6.** The region enclosed by the curves $y = \sqrt{\dfrac{x+1}{x^2+1}}$, $x=0$, $x=1$ and $y=0$ is rotated about the $x$-axis through $2\pi$ radians. Show that the volume of the solid thus generated is $\dfrac{\pi}{4}(\pi + \ln 4)$.




The volume generated $= \displaystyle\int_0^1 \pi \left(\sqrt{\frac{x+1}{x^2+1}}\right)^2 dx$ 5

$= \pi \left( \displaystyle\int_0^1 \frac{x}{x^2+1}\, dx + \int_0^1 \frac{1}{x^2+1}\, dx \right)$ 5

$= \pi \left( \frac{1}{2} \ln (x^2+1) \Big|_0^1 + \tan^{-1} x \Big|_0^1 \right)$ 5 + 5

$= \pi \left( \frac{1}{2} \ln 2 + \frac{\pi}{4} \right)$

$= \frac{\pi}{4} \left( \ln 4 + \pi \right)$ 5

$\boxed{25}$

**7.** Let $C$ be the parabola parametrically given by $x = at^2$ and $y = 2at$ for $t \in \mathbb{R}$, where $a \neq 0$. Show that the equation of the normal line to the parabola $C$ at the point $(at^2, 2at)$ is given by $y + tx = 2at + at^3$.

The normal line at the point $P \equiv (4a, 4a)$ on the parabola $C$ meets this parabola again at a point $Q \equiv (aT^2, 2aT)$. Show that $T = -3$.

$x = at^2$ , $y = 2at$

$\dfrac{dx}{dt} = 2at$ , $\dfrac{dy}{dt} = 2a$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = 2a \cdot \dfrac{1}{2at} = \dfrac{1}{t}$ for $t \neq 0$. 5

$\therefore$ The slope of the normal line $= -t$

The equation of the normal at $(at^2, 2at)$ is

$y - 2at = -t\ (x - at^2)$

$y + tx = 2at + at^3$ 5 (This is valid for $t = 0$ also.)

$P \equiv (4a, 4a)$ on $C \Rightarrow t = 2$.

The normal line at $P$ : $y + 2x = 4a + 8a = 12a$ 5

Since it meets $C$ at $(aT^2, 2aT)$, we have

$2aT + 2aT^2 = 12a$. 5

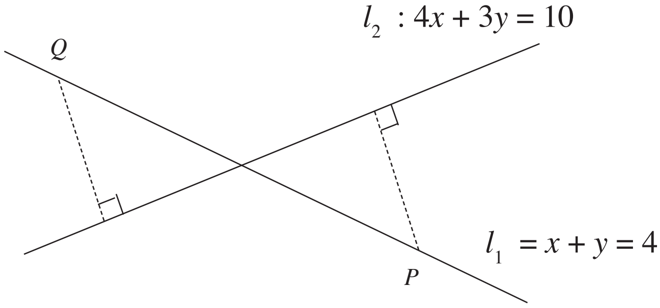
$\Leftrightarrow T^2 + T - 6 = 0 \Leftrightarrow (T - 2)(T + 3) = 0$

$\Leftrightarrow T = 2$ or $T = -3$

$\therefore T = -3$ 5

$\boxed{25}$

**8.** Let $l_1$ and $l_2$ be the straight lines given by $x+y=4$ and $4x+3y=10$, respectively. Two distinct points $P$ and $Q$ are on the line $l_1$ such that the perpendicular distance from each of these points to the line $l_2$ is 1 unit. Find the coordinates of $P$ and $Q$.




Any point on the line $l_1$ can be written in the form

$(t, 4-t),\ t \in \mathbb{R}$. 5

Let $P \equiv (t_1, 4-t_1)$

Perpendicular distance from $P$ to $l_2$ $= \dfrac{|4t_1 + 3(4-t_1) - 10|}{\sqrt{4^2+3^2}} = 1$ 5

$\therefore\ |t_1 + 2| = 5$

$\therefore\ t_1 = -7$ or $t_1 = 3$ 5

The coordinates of $P$ and $Q$ are

$(-7, 11)$ and $(3, 1)$. 5 + 5

**25**

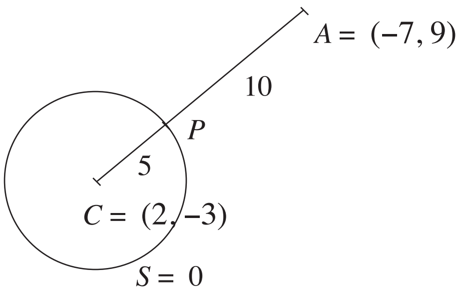
**9.** Show that the point $A \equiv (-7, 9)$ lies outside the circle $S \equiv x^2 + y^2 - 4x + 6y - 12 = 0$.
Find the coordinates of the point on the circle $S = 0$ nearest to the point $A$.

The centre $C$ of $S = 0$ is $(2, -3)$. 5

The radius $R$ of $S = 0$ is $\sqrt{4 + 9 + 12} = \sqrt{25} = 5$. 5

$CA^2 = 9^2 + 12^2 = 15^2 \Rightarrow CA = 15 > R = 5$. 5

$\therefore$ Point $A$ lies outside the given circle.




The point on the circle $S = 0$ nearest to point $A$ is the point $P$ at which $CA$ meets $S = 0$.

Note that $CP : PA = 5 : 10$

$= 1 : 2$ 5

$$\therefore \; P \equiv \left( \frac{2 \times 2 + 1\,(-7)}{3}, \; \frac{2\,(-3) + 1 \times 9}{3} \right)$$

i.e. $P \equiv (-1, 1)$ 5

**25**

**10.** Let $t = \tan\frac{\theta}{2}$ for $\theta \neq (2n+1)\pi$, where $n \in \mathbb{Z}$. Show that $\cos\theta = \frac{1-t^2}{1+t^2}$.

**Deduce** that $\tan\frac{\pi}{12} = 2-\sqrt{3}$.

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \quad 5$$

$$= \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} \quad 5 \qquad \text{for } \theta \neq (2n+1)\pi.$$

$$= \frac{1-t^2}{1+t^2}$$

Let $\theta = \frac{\pi}{6}$. Then $\frac{\sqrt{3}}{2} = \frac{1-t^2}{1+t^2}$

5

$$\Rightarrow \sqrt{3}\,(1+t^2) = 2\,(1-t^2)$$

$$(2+\sqrt{3})\,t^2 = 2-\sqrt{3}$$

$$\therefore \quad t^2 = \frac{(2-\sqrt{3})}{(2+\sqrt{3})} \quad 5$$

$$= (2-\sqrt{3})^2$$

$$\Rightarrow t = \tan\frac{\pi}{12} = 2-\sqrt{3} \quad 5 \qquad \left(\because \tan\frac{\pi}{12} > 0\right)$$

**25**

**11.** (*a*) Let $p \in \mathbb{R}$ and $0 < p \leq 1$. Show that 1 is **not** a root of the equation $p^2x^2 + 2x + p = 0$.

Let $\alpha$ and $\beta$ be the roots of this equation. Show that $\alpha$ and $\beta$ are both real.
Write down $\alpha + \beta$ and $\alpha\beta$ in terms of $p$, and show that

$$\frac{1}{(\alpha - 1)} \cdot \frac{1}{(\beta - 1)} = \frac{p^2}{p^2 + p + 2}.$$

Show also that the quadratic equation whose roots are $\frac{\alpha}{\alpha - 1}$ and $\frac{\beta}{\beta - 1}$ is given by $(p^2 + p + 2)x^2 - 2(p+1)x + p = 0$ and that both of these roots are positive.

(*b*) Let $c$ and $d$ be two **non-zero** real numbers and let $f(x) = x^3 + 2x^2 - dx + cd$. It is given that $(x - c)$ is a factor of $f(x)$ and that the remainder when $f(x)$ is divided by $(x - d)$ is $cd$. Find the values of $c$ and $d$.

For these values of $c$ and $d$, find the remainder when $f(x)$ is divided by $(x + 2)^2$.

(*a*) Suppose that 1 is a root of $p^2x^2 + 2x + p = 0$.

By substituting $x = 1$, we must have $p^2 + 2 + p = 0$. 5

This is impossible, as $p > 0$ implies that $p^2 + 2 + p > 0$. 5

$\therefore$ 1 is not a root of $p^2x^2 + 2x + p = 0$ $\boxed{10}$

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The discriminant $\Delta = 2^2 - 4p^2 \cdot p$ 10

$= 4(1 - p^3)$

$\geq 0 \quad (\because 0 < p \leq 1)$ 5

$\therefore$ $\alpha$ and $\beta$ are both real. 5 $\boxed{20}$

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$\alpha + \beta = -\frac{2}{p^2}$ and $\alpha\beta = \frac{1}{p}$ 5 + 5

Now,

$$\frac{1}{(\alpha - 1)} \cdot \frac{1}{(\beta - 1)} = \frac{1}{(\alpha\beta - (\alpha + \beta) + 1)}$$ 5

$$= \frac{1}{\frac{1}{p} + \frac{2}{p^2} + 1}$$

$$= \frac{p^2}{p^2 + p + 2}.$$ 5 $\boxed{20}$

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Now

$$\frac{\alpha}{\alpha-1} + \frac{\beta}{\beta-1} = \frac{\alpha(\beta-1)+\beta(\alpha-1)}{(\alpha-1)\,(\beta-1)}$$

$$= \frac{2\alpha\beta-(\alpha+\beta)}{(\alpha-1)\,(\beta-1)} \quad 5$$

$$= \left(\frac{2}{p} + \frac{2}{p^2}\right) \cdot \frac{p^2}{p^2+p+2} \quad 5$$

$$= \frac{2(p+1)}{p^2} \cdot \frac{p^2}{p^2+p+2}$$

$$= \frac{2(p+1)}{p^2+p+2} \quad 5$$

and

$$\frac{\alpha}{\alpha-1} \cdot \frac{\beta}{\beta-1} = \frac{\alpha\beta}{(\alpha-1)\,(\beta-1)}$$

$$= \frac{1}{p} \cdot \frac{p^2}{p^2+p+2}$$

$$= \frac{p}{p^2+p+2}\,. \quad 5$$

Hence, the required quadratic equation is given by

$$x^2 - \frac{2(p+1)}{p^2+p+2}\,x + \frac{p}{p^2+p+2} = 0 \quad 10$$

$$\Rightarrow \quad (p^2+p+2)\,x^2 - 2\,(p+1)\,x + p = 0 \quad 5$$

$\boxed{35}$

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Moreover, note that $\frac{\alpha}{(\alpha-1)}$ and $\frac{\beta}{(\beta-1)}$ are both real,

$$\frac{\alpha}{(\alpha-1)} + \frac{\beta}{(\beta-1)} = \frac{2(p+1)}{p^2+p+2} > 0, \quad (\because \; p > 0)\,,$$

5

and $$\frac{\alpha}{(\alpha-1)} \cdot \frac{\beta}{(\beta-1)} = \frac{p}{p^2+p+2} > 0, \quad (\because \; p > 0)\,.$$

Hence, both of these roots are possitive. 5

$\boxed{10}$

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(**b**) $f(x) = x^3 + 2x^2 - dx + cd$

Since $(x - c)$ is a factor, $f(c) = 0$. 5

$\Rightarrow c^3 + 2c^2 - dc + cd = 0$ 5

$\Rightarrow c^2 (c + 2) = 0$

$\Rightarrow c = -2 \quad (\because c \neq 0)$ 5

Since, when $f(x)$ is divided by $(x - d)$, the remainder is $cd$, we have

$$f(d) = cd.$$ 5

$\Rightarrow d^3 + 2d^2 - d^2 + cd = cd$ 5

$\Rightarrow d^3 + d^2 = 0$

$\Rightarrow d^2 (d + 1) = 0$

$\Rightarrow d = -1 \quad (\because d \neq 0)$ 5

$\therefore c = -2$ and $d = -1$. **30**

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$f(x) = x^3 + 2x^2 + x + 2$.

Let $Ax + B$ be the remainder, when $f(x)$ is divided by $(x + 2)^2$.

Then $f(x) \equiv (x + 2)^2 Q(x) + (Ax + B)$, where $Q(x)$ is a polynomial of degree 1.

So, $x^3 + 2x^2 + x + 2 \equiv (x + 2)^2 Q(x) + Ax + B$. 5

Substituting $x = -2$, we obtain $0 = -2A + B$. 5

By differentiating, we have

$$3x^2 + 4x + 1 = (x + 2)^2 Q'(x) + 2Q(x)(x + 2) + A.$$ 5

Again by substituting $x = -2$, we obtain

$$12 - 8 + 1 = A$$ 5

$\therefore A = 5$ and $B = 10$

Hence the remainder is $5x + 10$. 5 **25**

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Aliter

By long division we have,

$$\begin{array}{r|l} & x - 2 \\ \hline x^2 + 4x + 4 & x^3 + 2x^2 + x + 2 \\ & \underline{x^3 + 4x^2 + 4x} \\ & \quad -2x^2 - 3x + 2 \\ & \quad \underline{-2x^2 - 8x - 8} \\ & \qquad\quad 5x + 10. \end{array}$$

15

$$x^3 + 2x^2 + x + 2 \equiv (x^2 + 4x + 4)(x - 2) + (5x + 10)$$

$\therefore$ Required remainder is $5x + 10$. 10

$\boxed{25}$

**12.** (*a*) Let $P_1$ and $P_2$ be the two sets given by $\{A, B, C, D, E, 1, 2, 3, 4\}$ and $\{F, G, H, I, J, 5, 6, 7, 8\}$ respectively. It is required to form a password consisting of 6 elements taken from $P_1 \cup P_2$ of which 3 are different letters and 3 are different digits. In each of the following cases, find the number of different such passwords that can be formed:

(i) all 6 elements are chosen only from $P_1$,

(ii) 3 elements are chosen from $P_1$ and the other 3 elements from $P_2$.

(*b*) Let $U_r = \dfrac{1}{r(r+1)(r+3)(r+4)}$ and $V_r = \dfrac{1}{r(r+1)(r+2)}$ for $r \in \mathbb{Z}^+$.

Show that $V_r - V_{r+2} = 6U_r$ for $r \in \mathbb{Z}^+$.

**Hence**, show that $\displaystyle\sum_{r=1}^{n} U_r = \frac{5}{144} - \frac{(2n+5)}{6(n+1)(n+2)(n+3)(n+4)}$ for $n \in \mathbb{Z}^+$.

Let $W_r = U_{2r-1} + U_{2r}$ for $r \in \mathbb{Z}^+$.

**Deduce** that $\displaystyle\sum_{r=1}^{n} W_r = \frac{5}{144} - \frac{(4n+5)}{24(n+1)(n+2)(2n+1)(2n+3)}$ for $n \in \mathbb{Z}^+$.

**Hence**, show that the infinite series $\displaystyle\sum_{r=1}^{\infty} W_r$ is convergent and find its sum.

(*a*) $P_1 = \{A, B, C, D, E, 1, 2, 3, 4\}$ and $P_2 = \{F, G, H, I, J, 5, 6, 7, 8\}$

(i) The number of different ways of choosing 3 different letters and 3 different digits from $P_1 = {}^{5}C_{3} \cdot {}^{4}C_{3}$ 10

Hence the number of passwords that can be formed by choosing all 6 elements from $P_1$

$= {}^{5}C_{3} \cdot {}^{4}C_{3} \cdot 6!$ 5

$= 28800$ 5

$\boxed{20}$

(ii)

| Different ways of selecting | | | | Number of Passwords | |
|---|---|---|---|---|---|
| from $P_1$ | | from $P_2$ | | | |
| Letters | Digits | Letters | Digits | | |
| 3 | – | – | 3 | ${}^{5}C_{3} \cdot {}^{4}C_{3} \cdot 6! = 28800$ | 10 |
| 2 | 1 | 1 | 2 | ${}^{5}C_{2} \cdot {}^{4}C_{1} \cdot {}^{5}C_{1} \cdot {}^{4}C_{2} \cdot 6! = 864000$ | 10 |
| 1 | 2 | 2 | 1 | ${}^{5}C_{1} \cdot {}^{4}C_{2} \cdot {}^{5}C_{2} \cdot {}^{4}C_{1} \cdot 6! = 864000$ | 10 |
| – | 3 | 3 | – | ${}^{4}C_{3} \cdot {}^{5}C_{3} \cdot 6! = 28800$ | 10 |

Hence, the number of different passwords that can be formed by choosing 3 elements

from $P_1$ and the other 3 elements from $P_2 =$ 28800 + 864000 + 864000 + 28800 = 1785600

10

$\boxed{50}$

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**(*b*)** $U_r = \dfrac{1}{r(r+1)(r+3)(r+4)}$ and $V_r = \dfrac{1}{r(r+1)(r+2)}$ ; $r \in \mathbb{Z}^+$ .

Then,

$$V_r - V_{r+2} = \frac{1}{r(r+1)(r+2)} - \frac{1}{(r+2)(r+3)(r+4)} \quad 5$$

$$= \frac{(r+3)(r+4) - r(r+1)}{r(r+1)(r+2)(r+3)(r+4)}$$

$$= \frac{6(r+2)}{r(r+1)(r+2)(r+3)(r+4)} \quad 5$$

$$= 6U_r \quad 5$$

$\boxed{15}$

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Now note that,

$$\begin{array}{lll}
r = 1; & 6U_1 & = V_1 - \cancel{V_3}, \\
r = 2; & 6U_2 & = V_2 - \cancel{V_4}, \\
r = 3; & 6U_3 & = \cancel{V_3} - V_5, \\
r = 4; & 6U_4 & = \cancel{V_4} - V_6, \\
\vdots & \vdots & \vdots \\
r = n-3; & 6U_{n-3} & = V_{n-3} - \cancel{V_{n-1}} \\
r = n-2; & 6U_{n-2} & = V_{n-2} - \cancel{V_n} \\
r = n-1; & 6U_{n-1} & = \cancel{V_{n-1}} - V_{n+1} \\
r = n; & 6U_n & = \cancel{V_n} - V_{n+2}
\end{array}$$

10

10

$$\therefore \quad 6\sum_{r=1}^{n} U_r = V_1 + V_2 - V_{n+1} - V_{n+2} \quad 10$$

$$= \frac{1}{6} + \frac{1}{24} - \frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+2)(n+3)(n+4)} \quad 5$$

$$= \frac{5}{24} - \frac{2n+5}{(n+1)(n+2)(n+3)(n+4)}$$

$$\therefore \sum_{r=1}^{n} U_r = \frac{5}{144} - \frac{2n+5}{6(n+1)(n+2)(n+3)(n+4)} \quad 5$$

$\boxed{40}$

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$$W_r = U_{2r-1} + U_{2r}, \quad r \in \mathbb{Z}^+.$$

$$\therefore \sum_{r=1}^{n} W_r = \sum_{r=1}^{n} (U_{2r-1} + U_{2r})$$

$$= \sum_{r=1}^{2n} U_r \quad 5$$

$$= \frac{5}{144} - \frac{4n+5}{6(2n+1)(2n+2)(2n+3)(2n+4)}$$

$$\therefore \sum_{r=1}^{n} W_r = \frac{5}{144} - \frac{4n+5}{24(n+1)(n+2)(2n+1)(2n+3)} \quad 5$$

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Note that,

$$\lim_{n\to\infty} \sum_{r=1}^{n} W_r = \lim_{n\to\infty} \left(\frac{5}{144} - \frac{4n+5}{24(n+1)(n+2)(2n+1)(2n+3)}\right) \quad 5$$

$$= \frac{5}{144} - \lim_{n\to\infty} \frac{4n+5}{24(n+1)(n+2)(2n+1)(2n+3)}$$

$$= \frac{5}{144} \quad 5$$

$\therefore \sum_{r=1}^{\infty} W_r$ is convergent and the sum is $\frac{5}{144}$. 5

$\boxed{15}$

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**13.** (*a*) Let $\mathbf{A} = \begin{pmatrix} a & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -a & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} b & -2 \\ -1 & b+1 \end{pmatrix}$ be matrices such that $\mathbf{AB}^{\mathrm{T}} = \mathbf{C}$, where $a, b \in \mathbb{R}$.

Show that $a = 2$ and $b = 1$.

Show also that, $\mathbf{C}^{-1}$ **does not** exist.

Let $\mathbf{P} = \frac{1}{2}(\mathbf{C} - 2\mathbf{I})$. Write down $\mathbf{P}^{-1}$ and find the matrix $\mathbf{Q}$ such that $2\mathbf{P}(\mathbf{Q} + 3\mathbf{I}) = \mathbf{P} - \mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(*b*) Let $z, z_1, z_2 \in \mathbb{C}$.

Show that (i) $\operatorname{Re} z \le |z|$, and

(ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ for $z_2 \ne 0$.

**Deduce** that $\operatorname{Re}\left(\dfrac{z_1}{z_1+z_2}\right) \le \dfrac{|z_1|}{|z_1+z_2|}$ for $z_1 + z_2 \ne 0$.

Verify that $\operatorname{Re}\left(\dfrac{z_1}{z_1+z_2}\right) + \operatorname{Re}\left(\dfrac{z_2}{z_1+z_2}\right) = 1$ for $z_1 + z_2 \ne 0$,

and show that $|z_1 + z_2| \le |z_1| + |z_2|$ for $z_1, z_2 \in \mathbb{C}$.

(*c*) Let $\omega = \frac{1}{2}\left(1 - \sqrt{3}\,i\right)$.

Express $1+\omega$ in the form $r(\cos\theta + i\sin\theta)$; where $r(>0)$ and $\theta\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$ are constants to be determined.

Using De Moivre's theorem, show that $(1+\omega)^{10} + (1+\bar{\omega})^{10} = 243$.

(*a*) $$AB^T = \begin{pmatrix} a & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -a \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2a-3 & a-4 \\ -1 & a \end{pmatrix}$$

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$$AB^T = C \Leftrightarrow \begin{pmatrix} 2a-3 & a-4 \\ -1 & a \end{pmatrix} = \begin{pmatrix} b & -2 \\ -1 & b+1 \end{pmatrix}$$

$\Leftrightarrow$ $2a - 3 = b,$ $a - 4 = -2$ and $a = b + 1.$ 10

$\Leftrightarrow$ $a = 2$ and $b = 1$, (from any two equations above) and these values satisfy the remaining equation. 5

$\boxed{30}$

$$C = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = 0 \quad (5)$$

$\therefore$ $C^{-1}$ does not exist. (5)

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Aliter

For the existence of $C^{-1}$ :

there must exist $p, q, r, s \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (5)$$

$\Rightarrow p - 2r = 1, \; -p + 2r = 0, \; q - 2s = 0$ and $-q + 2s = 1$

This is a contradiction

$\therefore C^{-1}$ does not exist. (5)

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$$P = \frac{1}{2}(C - 2I) = \frac{1}{2}\left\{\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} -1 & -2 \\ -1 & 0 \end{pmatrix} \quad (5)$$

$$\Rightarrow P^{-1} = 2\left(\frac{1}{-2}\right)\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix} \quad (10)$$

$$2P(Q + 3I) = P - I$$

$$\Leftrightarrow \quad 2(Q + 3I) = I - P^{-1} \quad (5)$$

$$\therefore \quad 2(Q + 3I) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \quad (5)$$

$$\Rightarrow Q = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} - 3I$$

$$= \begin{pmatrix} -\frac{5}{2} & 1 \\ \frac{1}{2} & -3 \end{pmatrix} \quad (5)$$

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**(*b*)** $z,\ z_1,\ z_2 \in \mathbb{C}$.

(i) Let $z = x + iy,\quad x, y \in \mathbb{R}$.

$\text{Re}\, z = x \leq \sqrt{x^2 + y^2} = |z|$ 5

(ii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1) \times (\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2) \times (\cos\theta_2 - i\sin\theta_2)} = \frac{r_1}{r_2}\ \frac{\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]}{1}$$

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$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$ 5

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$$\text{Re}\left(\frac{z_1}{z_1 + z_2}\right) \leq \left|\frac{z_1}{z_1 + z_2}\right| = \frac{|z_1|}{|z_1 + z_2|} \quad ; \text{for } z_1 + z_2 \neq 0.$$

5 by (i) 5 by (ii)

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For $z_1 + z_2 \neq 0$, we have

$$\frac{z_1}{z_1 + z_2} + \frac{z_2}{z_1 + z_2} = 1$$ 5

$$\text{Re}\left(\frac{z_1}{z_1 + z_2} + \frac{z_2}{z_1 + z_2}\right) = 1$$

$$\text{Re}\left(\frac{z_1}{z_1 + z_2}\right) + \text{Re}\left(\frac{z_2}{z_1 + z_2}\right) = 1$$ 5

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$$\Rightarrow \quad 1 = \mathrm{Re}\left(\frac{z_1}{z_1+z_2}\right) + \mathrm{Re}\left(\frac{z_2}{z_1+z_2}\right) \le \left|\frac{z_1}{z_1+z_2}\right| + \left|\frac{z_2}{z_1+z_2}\right| \quad \text{by (i)} \quad 5$$

$$= \frac{|z_1|}{|z_1+z_2|} + \frac{|z_2|}{|z_1+z_2|} \quad \text{by (ii)}$$

$$= \frac{|z_1|+|z_2|}{|z_1+z_2|} \quad 5$$

$$\Rightarrow \quad |z_1+z_2| \le |z_1| + |z_2| \qquad (\because |z_1+z_2| > 0)$$

Now if $z_1 + z_2 = 0$, then

$$|z_1+z_2| = 0 \le |z_1| + |z_2|$$

Hence, the result is true for all $z_1, z_2 \in \mathbb{C}$.

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(*c*) $\omega = \frac{1}{2}(1 - \sqrt{3}\, i)$

$$1 + \omega = \sqrt{3}\left[\frac{\sqrt{3}}{2} + i\left(\frac{-1}{2}\right)\right] = r(\cos\theta + i\sin\theta), \quad 5$$

where $r = \sqrt{3}$ and $\theta = -\frac{\pi}{6}$. 5

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$$(1+\omega)^{10} = (\sqrt{3})^{10}\left[\cos(10\theta) + i\sin(10\theta)\right] \text{ by De Moivre's theorem} \quad 5$$

$$1 + \overline{\omega} = \overline{1+\omega} = \sqrt{3}(\cos\theta - i\sin\theta) = \sqrt{3}\left[\cos(-\theta) + i\sin(-\theta)\right]$$

$$\Rightarrow (1+\overline{\omega})^{10} = \left(\sqrt{3}\right)^{10}\left[(\cos(-10\theta) + i\sin(-10\theta)\right] \quad 5$$

$$\therefore (1+\omega)^{10} + (1+\overline{\omega})^{10} = \left(\sqrt{3}\right)^{10} \times 2\cos(10\theta) \quad 5$$

$$= 3^5 \times 2 \times \frac{1}{2}$$

$$= 243. \quad 5$$

**20**

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**14.** (*a*) Let $f(x) = \dfrac{9(x^2 - 4x - 1)}{(x-3)^3}$ for $x \neq 3$.

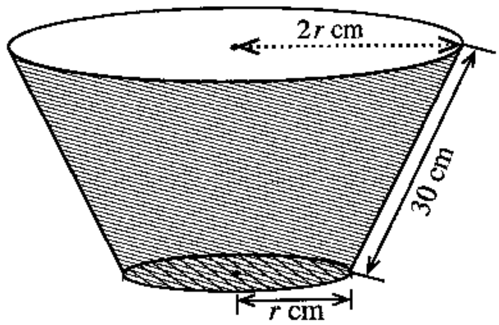
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = -\dfrac{9(x+3)(x-5)}{(x-3)^4}$ for $x \neq 3$.

Sketch the graph of $y=f(x)$ indicating the asymptotes, $y$-intercept and the turning points.

It is given that $f''(x) = \dfrac{18(x^2-33)}{(x-3)^5}$ for $x \neq 3$.

Find the $x$-coordinates of the points of inflection of the graph of $y=f(x)$.

(*b*) The adjoining figure shows a basin in the form of a frustum of a right circular cone with a bottom. The slant length of the basin is 30 cm and the radius of the upper circular edge is twice the radius of the bottom. Let the radius of the bottom be $r$ cm.




Show that the volume $V$ cm$^3$ of the basin is given by $V = \dfrac{7}{3}\pi r^2\sqrt{900 - r^2}$ for $0<r<30$.

Find the value of $r$ such that volume of the basin is maximum.

(*a*) For $x \neq 3$; $f(x) = \dfrac{9(x^2 - 4x - 1)}{(x-3)^3}$

Then

$$f'(x) = 9\left[\frac{1}{(x-3)^3}(2x-4) - \frac{3(x^2-4x-1)}{(x-3)^4}\right] \quad 20$$

$$= 9\left[\frac{2x^2 - 10x + 12 - 3(x^2 - 4x - 1)}{(x-3)^4}\right]$$

$$= \frac{-9(x+3)(x-5)}{(x-3)^4} \quad \text{for } x \neq 3 \quad 5$$

$\boxed{25}$

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Horizontal asymptotes : $\lim\limits_{x \to \pm\infty} f(x) = 0 \quad \therefore\ y = 0.$ 5

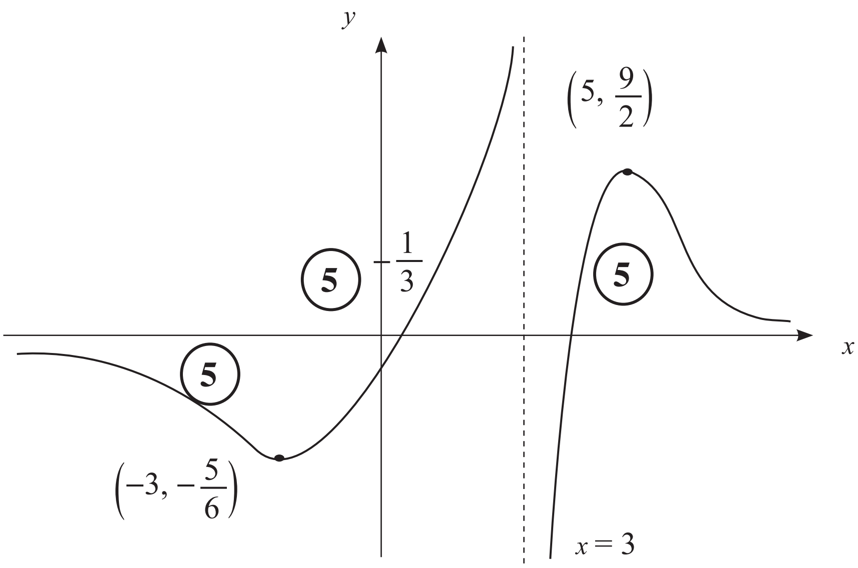
$\lim\limits_{x \to 3^-} f(x) = \infty$ and $\lim\limits_{x \to 3^+} f(x) = -\infty.$

Vertical asymptote : $x = 3$. 5

At the turning points $f'(x) = 0. \Leftrightarrow x = -3$ or $x = 5$. 5

| | $-\infty < x < -3$ | $-3 < x < 3$ | $3 < x < 5$ | $5 < x < \infty$ |
|---|---|---|---|---|
| sign of $f^{/}(x)$ | $(-)$ | $(+)$ | $(+)$ | $(-)$ |
| | 5 | 5 | 5 | 5 |
| $f(x)$ is | $\searrow$ | $\nearrow$ | $\nearrow$ | $\searrow$ |

There are two turning points: $\left(-3, -\frac{5}{6}\right)$ 5 is a local minimum and $\left(5, \frac{9}{2}\right)$ 5 is a local maximum.




**60**

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For $x \neq 3$;

$$f^{//}(x) = \frac{18\,(x - \sqrt{33}\,)\,(x + \sqrt{33})}{(x-3)^5} \;.$$

$$f^{//}(x) = 0 \Leftrightarrow x = \pm\sqrt{33}\,. \quad 5$$

| | $-\infty < x < -\sqrt{33}$ | $-\sqrt{33} < x < 3$ | $3 < x < \sqrt{33}$ | $\sqrt{33} < x < \infty$ |
|---|---|---|---|---|
| sign of $f^{//}(x)$ | $(-)$ | $(+)$ | $(-)$ | $(+)$ |
| concavity | concave down | concave up | concave down | concave up |

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$\therefore$ There are two inflection points:

$x = -\sqrt{33}$ and $x = \sqrt{33}$ are the $x-$ coordinates of the points of inflection. 5

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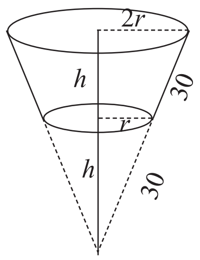
(**b**)




For $0 < r < 30$;

$h = \sqrt{900 - r^2}$ 5

The volume $V$ is given by

$$V = \frac{1}{3}\pi(2r)^2 \times 2h - \frac{1}{3}\pi r^2 h \quad 5$$

$$= \frac{7}{3}\pi r^2 h$$

$$= \frac{7}{3}\pi r^2 \sqrt{900 - r^2}\,. \quad 5$$

15

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For $0 < r < 30$,

$$\frac{dV}{dr} = \frac{7}{3}\pi\left[2r\sqrt{900 - r^2} + r^2\frac{(-2r)}{2\sqrt{900 - r^2}}\right] \quad 5$$

$$= \frac{7}{3}\pi\,\frac{[2r(900 - r^2) - r^3]}{\sqrt{900 - r^2}}$$

$$= 7\pi r\,\frac{(600 - r^2)}{\sqrt{900 - r^2}}\,. \quad 5$$

$$\frac{dV}{dr} = 0 \Leftrightarrow r = 10\sqrt{6} \quad (\because r > 0) \quad 5$$

For $0 < r < 10\sqrt{6}$, $\frac{dV}{dr} > 0$ and for $r > 10\sqrt{6}$, $\frac{dV}{dr} < 0$

5 5

$\therefore$ $V$ is maximum when $r = 10\sqrt{6}$. 5

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15. (*a*) Using the substitution $x = 2\sin^2\theta + 3$ for $0 \le \theta \le \frac{\pi}{4}$, evaluate $\int_3^4 \sqrt{\frac{x-3}{5-x}}\,dx$.

(*b*) Using partial fractions, find $\int \frac{1}{(x-1)(x-2)}\,dx$.

Let $f(t) = \int_3^t \frac{1}{(x-1)(x-2)}\,dx$ for $t > 2$.

**Deduce** that $f(t) = \ln(t-2) - \ln(t-1) + \ln 2$ for $t > 2$.

Using integration by parts, find $\int \ln(x-k)\,dx$, where $k$ is a real constant.
**Hence**, find $\int f(t)\,dt$.

(*c*) Using the formula $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, where $a$ and $b$ are constants,

show that $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x}\,dx = \int_{-\pi}^{\pi} \frac{e^x \cos^2 x}{1+e^x}\,dx$.

**Hence**, find the value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x}\,dx$.

(*a*) For $0 \le \theta \le \frac{\pi}{4}$ :

$x = 2\sin^2\theta + 3 \Rightarrow dx = 4\sin\theta\cos\theta\ d\theta$ 5

$x = 3 \Leftrightarrow 2\sin^2\theta = 0 \Leftrightarrow \theta = 0$ 5

$x = 4 \Leftrightarrow 2\sin^2\theta = 1 \Leftrightarrow \sin\theta = \frac{1}{\sqrt{2}} \Leftrightarrow \theta = \frac{\pi}{4}$ 5

Then $\int_3^4 \sqrt{\frac{x-3}{5-x}}\ dx = \int_0^{\frac{\pi}{4}} \sqrt{\frac{2\sin^2\theta}{2-2\sin^2\theta}} \cdot 4\sin\theta\cos\theta\ d\theta$ 5

$= \int_0^{\frac{\pi}{4}} 4\sin^2\theta\ d\theta$ 5

$= 2\int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\ d\theta$ 5

$= 2\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{4}}$ 5

$= \frac{\pi}{2} - 1$ 5

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**(b)** $$\frac{1}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

$$\Leftrightarrow 1 = A(x-2) + B(x-1) \text{ for } x \neq 1, 2.$$

Comparing coefficients of powers of $x$ :

$x^1$ : $A + B = 0$ 5

$x^0$ : $-2A - B = 1$ 5

$A = -1$ and $B = 1$ 5

Then $$\int \frac{1}{(x-1)(x-2)} \, dx = \int \frac{-1}{(x-1)} \, dx + \int \frac{1}{(x-2)} \, dx$$ 10

$= \ln|x-2| - \ln|x-1| + C$, where $C$ is an arbitrary constant.

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$$f(t) = \int_3^t \frac{1}{(x-1)(x-2)} \, dx$$

$$= \left(\ln|x-2| - \ln|x-1|\right)\Big|_3^t$$ 5

$$= \ln(t-2) - \ln(t-1) + \ln 2 \text{ for } t > 2.$$ 5

**10**

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$$\int \ln(x-k) \, dx = x\ln(x-k) - \int \frac{x}{(x-k)} \, dx$$ 5

$$= x\ln(x-k) - \int 1 \, dx - \int \frac{k}{(x-k)} \, dx$$ 5

$$= x\ln(x-k) - x - k\ln(x-k) + C$$ 5

$= (x-k)\ln(x-k) - x + C$, where $C$ is an arbitrary constant.

**15**

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$$\int f(t) \, dt = \int \ln(t-2) \, dt - \int \ln(t-1) \, dt + \int \ln 2 \, dt$$ 5

$$= (t-2)\ln(t-2) - t - \Big[(t-1)\ln(t-1) - t\Big] + t\ln 2 + D$$

$= (t-2)\ln(t-2) - (t-1)\ln(t-1) + t\ln 2 + D$, where $D$ is an arbitrary constant.

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**(c)** Using the formula $\displaystyle\int_a^b f(x)\ dx = \int_a^b (a+b-x)\ dx,$

$$\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x}\ dx = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+e^{-x}}\ dx \quad 5$$

$$= \int_{-\pi}^{\pi} \frac{e^x \cos^2 x}{1+e^x}\ dx \quad 5$$

$$\boxed{10}$$

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$$2\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x}\ dx = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^{-x}}\ dx + \int_{-\pi}^{\pi} \frac{e^x \cos^2 x}{1+e^x}\ dx \quad 5$$

$$= \int_{-\pi}^{\pi} \frac{(1+e^x)\cos^2 x}{(1+e^x)}\ dx$$

$$= \int_{-\pi}^{\pi} \cos^2 x\ dx \quad 5$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos 2x)\ dx \quad 5$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_{-\pi}^{\pi} \quad 5$$

$$\therefore \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x} = \frac{\pi}{2} \quad 5$$

$$\boxed{25}$$

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**16.** Write down the coordinates of the point of intersection $A$ of the straight lines $12x-5y-7=0$ and $y=1$.

Let $l$ be the bisector of the acute angle formed by these lines. Find the equation of the straight line $l$.

Let $P$ be a point on $l$. Show that the coordinates of $P$ can be written as $(3\lambda+1,\ 2\lambda+1)$, where $\lambda\in\mathbb{R}$.

Let $B\equiv(6,0)$. Show that the equation of the circle with the points $B$ and $P$ as ends of a diameter can be written as $S+\lambda U=0$, where $S\equiv x^2+y^2-7x-y+6$ and $U\equiv -3x-2y+18$.

**Deduce** that $S=0$ is the equation of the circle with $AB$ as a diameter.

Show that $U=0$ is the equation of the straight line through $B$, perpendicular to $l$.

Find the coordinates of the fixed point which is distinct from $B$, and lying on the circles with the equation $S+\lambda U=0$ for all $\lambda\in\mathbb{R}$.

Find the value of $\lambda$ such that the circle given by $S=0$ is orthogonal to the circle given by $S+\lambda U=0$.

$12x-5y-7=0$ and $y=1 \Rightarrow x=1,\quad y=1$

$\therefore\ A\equiv(1,1)$ 10

$\boxed{10}$

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Equations of the bisectors are given by

$$\frac{12x-5y-7}{13}=\pm\frac{(y-1)}{1}$$ 10

$\Rightarrow\ 12x-5y-7=13(y-1)$ or $12x-5y-7=-13(y-1)$

$\Rightarrow\ 2x-3y+1=0$ or $3x+2y-5=0$ 5 + 5

The angle $\theta$ between $y=1$ and $2x-3y+1=0$, is given by

$$\tan\theta=\left|\frac{\frac{2}{3}-0}{1+\frac{2}{3}(0)}\right|=\frac{2}{3}<1$$ 5

$\therefore\ l:\ 2x-3y+1=0.$ 5

$\boxed{30}$

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Note that for a point $(x, y)$ on $l$ ;

$$\frac{(x-1)}{3} = \frac{(y-1)}{2} = \lambda \text{ (say)} \quad 5$$

$\Rightarrow x = 3\lambda + 1, \quad y = 2\lambda + 1.$ 5

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$\therefore \; P \equiv (3\lambda + 1,\ 2\lambda + 1), \quad \lambda \in \mathbb{R}.$

Note that $B \equiv (6, 0)$ and $P \equiv (3\lambda + 1,\ 2\lambda + 1)$

$\therefore$ Equation of the circle with $BP$ as a diameter is given by

$(x-6)\ (x - (3\lambda + 1)) + (y-0)\ (y - (2\lambda + 1)) = 0$ 10

i.e. $(x^2 + y^2 - 7x - y + 6) + \lambda(-3x - 2y + 18) = 0$ 5

This is of the form $S + \lambda U = 0$, where $S \equiv x^2 + y^2 - 7x - y + 6$ and $U \equiv -3x - 2y + 18$.

5 5 25

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$S = 0$ corresponds to $\lambda = 0$. $\Rightarrow P = (1, 1) \equiv A.$ 5

$\therefore \; S = 0$ is the equation of the circle with $AB$ as a diameter. 5

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Since the slope of $l$ is $\frac{2}{3}$ , the equation of the line perpendicular to $l$ passing through $B$ is $3x + 2y + \mu = 0$, $\mu$ to be determined. 10

Since $B$ lies on $3x + 2y + \mu = 0$, we have $18 + \mu = 0 \Rightarrow \mu = -18.$ 5

$\therefore$ Required equation is $3x + 2y - 18 = 0$ 5

i.e. $U = -3x - 2y + 18 = 0.$

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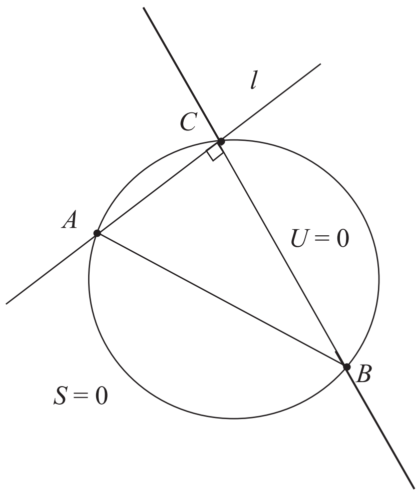
$\lambda \in \mathbb{R}$, $S + \lambda U = 0$ passes through the intersection point of $S = 0$ and $U = 0$ 10

One of these points is $B$ and the other point $C$ is the intersection point of $l$ and $U = 0$. 10




$\therefore$ The coordinates of $C$ is given by

$$u \equiv -3x - 2y + 18 = 0$$

and $l \equiv 2x - 3y + 1 = 0$

$\Rightarrow$ $x = 4$ and $y = 3$

$\therefore$ $C \equiv (4, 3)$ . 5

**25**

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The circles ;

$S = 0$ and $S + \lambda U = 0$ are orthogonal

$$\Leftrightarrow \quad 2\left(-\frac{1}{2}(3\lambda + 7)\right)\left(-\frac{7}{2}\right) + 2\left(-\frac{1}{2}(2\lambda + 1)\right)\left(-\frac{1}{2}\right) = 6 + 18\lambda + 6$$

5 5 5

$\Leftrightarrow$ $13\lambda = 26$ 5

$\Leftrightarrow$ $\lambda = 2$.

**20**

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**17.** (*a*) Write down $\sin(A+B)$ in terms of $\sin A$, $\cos A$, $\sin B$ and $\cos B$, and obtain a similar expression for $\sin(A-B)$.

**Deduce** that

$$2\sin A\ \cos B = \sin(A+B)+\sin(A-B) \text{ and}$$
$$2\cos A\ \sin B = \sin(A+B)-\sin(A-B).$$

**Hence**, solve $2\sin 3\theta\cos 2\theta = \sin 7\theta$ for $0<\theta<\frac{\pi}{2}$.

(*b*) In a triangle $ABC$, the point $D$ lies on $AC$ such that $BD = DC$ and $AD = BC$. Let $B\hat{A}C = \alpha$ and $A\hat{C}B = \beta$. Using the Sine Rule for suitable triangles, show that $2\sin\alpha\cos\beta = \sin(\alpha+2\beta)$.

If $\alpha:\beta = 3:2$, using the last result in (*a*) above, show that $\alpha = \frac{\pi}{6}$.

(*c*) Solve $2\tan^{-1}x + \tan^{-1}(x+1) = \frac{\pi}{2}$. **Hence**, show that $\cos\left(\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right) = \frac{3}{\sqrt{10}}$.

(*a*) $\sin(A+B) = \sin A\cos B + \cos A\sin B$ ——— (1) 5

Now $\sin(A-B) = \sin(A+(-B))$ 5

$= \sin A\cos(-B) + \cos A\sin(-B)$

$\therefore\ \sin(A-B) = \sin A\cos B - \cos A\sin B$ ——— (2) 5

$\boxed{15}$

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(1) + (2) $\Rightarrow \sin(A+B) + \sin(A-B) = 2\sin A\cos B,$ 5

(1) − (2) $\Rightarrow \sin(A+B) - \sin(A-B) = 2\cos A\sin B.$ 5

$\boxed{10}$

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$\boxed{0<\theta<\frac{\pi}{2}.}$

$2\sin 3\theta\cos 2\theta = \sin 7\theta,$

$\Leftrightarrow \sin 5\theta + \sin\theta = \sin 7\theta$ 5

$\Leftrightarrow \sin 7\theta - \sin 5\theta - \sin\theta = 0$

$\Leftrightarrow \sin(6\theta+\theta) - \sin(6\theta-\theta) - \sin\theta = 0$

$\Leftrightarrow 2\cos 6\theta\sin\theta - \sin\theta = 0$ 5

$\Leftrightarrow \sin\theta(2\cos 6\theta - 1) = 0$

$\Leftrightarrow \cos 6\theta = \frac{1}{2}$ since $0<\theta<\frac{\pi}{2}$, $\sin\theta > 0$

5

$$\Rightarrow \quad 6\theta = 2n\pi \pm \frac{\pi}{3}\;;\; n \in \mathbb{Z}. \quad 5 + 5$$

$$\Rightarrow \quad \theta = \frac{n\pi}{3} \pm \frac{\pi}{18}\;;\; n \in \mathbb{Z}.$$

$$\Rightarrow \quad \theta = \frac{\pi}{18},\ \frac{5\pi}{18},\ \frac{7\pi}{18},\ \left(\because\ 0 < \theta < \frac{\pi}{2}\right) \quad 5$$

$\boxed{30}$

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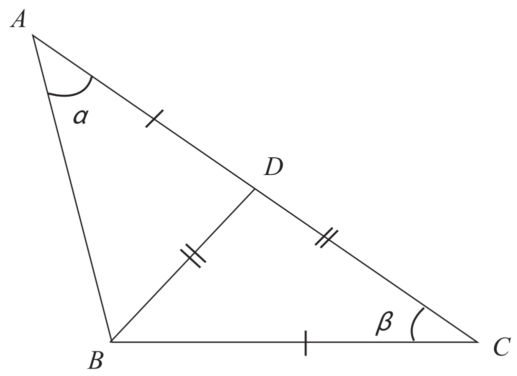
**(b)**




Note that

$C\hat{B}D = \beta,\ A\hat{D}B = 2\beta,$

and $A\hat{B}D = \pi - (\alpha + 2\beta)$

Using the sine Rule :

for the triangle $ABD$, we have

$$\frac{BD}{\sin B\hat{A}D} = \frac{AD}{\sin A\hat{B}D} \quad 10$$

$$\Rightarrow \quad \frac{BD}{\sin \alpha} = \frac{AD}{\sin(\pi - (\alpha + 2\beta))}$$

$$\Rightarrow \quad \frac{BD}{\sin \alpha} = \frac{AD}{\sin(\alpha + 2\beta)} \quad 5 \quad \text{———} \quad (1)$$

for the triangle $BDC$, we have

$$\frac{CD}{\sin D\hat{B}C} = \frac{BC}{\sin B\hat{D}C} \quad 10$$

$$\Rightarrow \quad \frac{CD}{\sin \beta} = \frac{BC}{\sin 2\beta} \quad \text{———} \quad (2) \quad 5$$

$\because\ BD = DC$ and $AD = BC$, from (1) and (2), we get

$$\frac{\sin \alpha}{\sin \beta} = \frac{\sin(\alpha + 2\beta)}{\sin 2\beta} \quad 5$$

$\Rightarrow \quad 2\sin\alpha\cos\beta = \sin(\alpha + 2\beta).$ 5

**40**

If $\alpha : \beta = 3 : 2$, then we have

$2\sin\alpha\cos\frac{2\alpha}{3} = \sin\frac{7\alpha}{3}$ 5

$\Rightarrow \quad 2\sin 3\left(\frac{\alpha}{3}\right)\cos 2\left(\frac{\alpha}{3}\right) = \sin 7\left(\frac{\alpha}{3}\right)$ 5

$\Rightarrow \quad \frac{\alpha}{3} = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}.$

$\Rightarrow \quad \alpha = \frac{\pi}{6}, \frac{15\pi}{18}, \frac{21\pi}{18}$ 5

$\because \; BC = AD < AC$, $\alpha$ must be an acute angle.

$\therefore \alpha = \frac{\pi}{6}.$ 5

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**(c)** $2\tan^{-1}x + \tan^{-1}(x+1) = \frac{\pi}{2}$

Let $\alpha = \tan^{-1}(x)$ and $\beta = \tan^{-1}(x+1)$. Note that $x \neq \pm 1$.

Then $2\alpha + \beta = \frac{\pi}{2}.$ 5

$\Leftrightarrow \quad 2\alpha = \frac{\pi}{2} - \beta$

$\Leftrightarrow \quad \tan 2\alpha = \tan\left(\frac{\pi}{2} - \beta\right)$ 5

$\Leftrightarrow \quad \frac{2\tan\alpha}{1-\tan^2\alpha} = \cot\beta$ 5 + 5

$\Leftrightarrow \quad \frac{2x}{1-x^2} = \frac{1}{x+1}$ 5

$\Leftrightarrow \quad 2x = 1 - x \quad (\because \; x \neq \pm 1)$

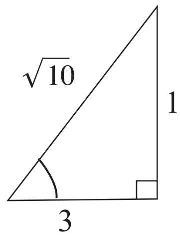
$\Leftrightarrow \quad x = \frac{1}{3}.$ 5

**25**

Note that

$$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{4}{3}\right) = \frac{\pi}{2}.$$

$$\Rightarrow \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{3}\right)$$




$$\Rightarrow \cos\left(\left(\frac{\pi}{4}\right) - \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right) = \cos\left(\tan^{-1}\left(\frac{1}{3}\right)\right)$$ 5

$$\therefore \cos\left(\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right) = \frac{3}{\sqrt{10}}$$ 5

**10**